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CYLINDRICAL SHELL LOADED BY RADIAL FORCES AROUND CIRCULAR REGIONS

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An approximate solution of the problem of a stressed-deformed state of an infinitely long cylindrical shell loaded by radial forces around circular regions is constructed by the method of asymptotic synthesis. To calculate normal displacement, tangential forces, and bending moments, expressions in the form of simple trigonometric series are suggested, by which the effect of the sizes and the quantity of loaded regions on the stressed-deformed state is studied.

Bending of a cylindrical panel by a normal force distributed around a circular region was considered in [1-3]. Here it was assumed that there was only one loaded region and it was situated at a considerable distance from the ends of the panel. The solution was constructed by the method of two-dimensional Fourier transform and then it was reduced to tabulated Thomson functions. In the case of a small radius of the region simple asymptotic formulas were suggested to calculate forces and moments.

This paper considers loading of a closed shell by a system of forces, i.e., around several circular regions whose centers are regularly located along the director circle of the cylinder. An approximate solution of the problem is constructed by the method of asymptotic synthesis (MAS) which is stated in [4, 5]. The solution is reduced to a single trigonometric series. Then, based on this solution numerical information giving an idea about the mutual effect of loaded regions on the value of inner force factors is analyzed.

Being guided by MAS, we represent a local stressed state caused by an external effect by a sum of two terms. The first refers to the basic state and the second expresses the so-called local end effect. We describe the basic state by a simplified equation of the semimomentless theory of shells written with respect to a resolution function $\Phi(\alpha, \beta)$ [4]:

$$\frac{\partial^4 \Phi}{\partial \alpha^4} + c^2 \frac{\partial^8 \Phi}{\partial \beta^8} = \frac{R^2}{Eh} p(\alpha, \beta), \quad c^2 = h^2/(12R^2(1-\nu^2)). \tag{1}$$

Radial displacement $w^{\mathrm{b}}(\alpha, \beta)$, tangential forces $T_1^{\mathrm{b}}(\alpha, \beta)$, $T_2^{\mathrm{b}}(\alpha, \beta)$ and bending moments $G_1^{\mathrm{b}}(\alpha, \beta)$, $G_2^{\mathrm{b}}(\alpha, \beta)$ of the basic state are expressed in terms of a resolution function by the expressions [5]:

$$w^{\mathrm{b}} = \frac{\partial^4 \Phi}{\partial \beta^4}; \quad T_1^{\mathrm{b}} = -\frac{Eh}{R} \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2}; \quad T_2^{\mathrm{b}} = 0;$$
$$G_1^{\mathrm{b}} = \nu G_2^{\mathrm{b}} = -\nu \frac{D}{R^2} \frac{\partial^6 \Phi}{\partial \beta^6}; \quad D = \frac{Eh^3}{12(1-\nu^2)} = Ehc^2R^2. \tag{2}$$

First we determine these quantities assuming that the outer load $p(\alpha, \beta)$ is a system of k radial forces (see Fig. 1) regularly concentrated along the director circle in section $\alpha = 0$. Taking one of the forces applied at the point with the coordinates (0; 0), we expand the load into a series in terms of cosines

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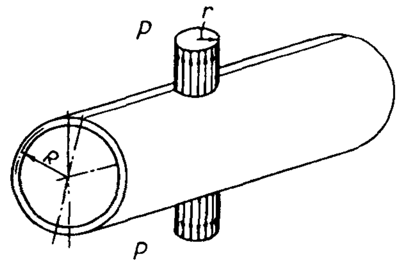

Fig. 1. Scheme of shell loading by radial forces P around two circular regions ($k = 2$) with radius $r = aR$.

$$p(\alpha, \beta) = R^{-1}\, \delta(\alpha - 0) \sum_{n=0}^{\infty} p_n \cos kn\beta, \tag{3}$$

where $p_0 = k/2\pi R$; $p_n = k/\pi R$; $n \in N$; $\delta(\alpha - 0)$ is the Dirac function.

Using (3), we construct the solution of Eq. (1), which decays at $|\alpha| \to \infty$. Having applied the Fourier cosine transform, we have (neglecting the zeroth harmonics)

$$\Phi(\alpha, \beta) = \frac{k}{\pi^2 Eh} \sum_{n=1}^{\infty} \cos kn\beta\, \Psi_n(\alpha, k).$$

Here

$$\Psi_n(\alpha, k) = \int_0^{\infty} \frac{\cos \lambda \alpha \, d\lambda}{\lambda^4 + c^2 (kn)^8}.$$

Having differentiated this solution in accordance with (2), we find the expressions for radial displacements and inner force factors of the basic state

$$w^{\mathrm{b}} = \frac{k^5}{\pi^2 Eh} \sum_{n=1}^{\infty} n^4 \cos kn\beta\, \Psi_n(\alpha, k);$$

$$T_1^{\mathrm{b}} = -\frac{k^3}{\pi^2 R} \sum_{n=1}^{\infty} n^2 \cos kn\beta \int_0^{\infty} \frac{\lambda^2 \cos \lambda\alpha}{\lambda^4 + c^2 (kn)^8}\, d\lambda; \tag{4}$$

$$G_1^{\mathrm{b}} = \nu G_2^{\mathrm{b}} = \frac{\nu k^7 c^2}{\pi^2} \sum_{n=1}^{\infty} n^6 \cos kn\beta\, \Psi_n(\alpha, k); \quad T_2^{\mathrm{b}} = 0.$$

To obtain values of these quantities at the center of one of the circular planes of a radius aR that is uniformly loaded by external force P, we integrate the right-hand sides of expressions (4) with respect to a circular region and then multiply the results of integration by $P/(\pi a^2 R^2)$. This transformation yields

$$w^{\mathrm{b}} = \frac{4Pk^5}{\pi^3 Eha^2} \sum_{n=1}^{\infty} n^4 S(k, n, a);$$

$$S(k, n, a) = \int_0^{a} \cos kn\beta \, d\beta \int_0^{\infty} \frac{\sin(\lambda \sqrt{a^2 - \beta^2})\, d\lambda}{\lambda(\lambda^4 + c^2 (kn)^8)};$$

$$T_1^{\mathrm{b}} = -\frac{4Pk^3}{\pi^3 Ra} \sum_{n=1}^{\infty} n^2 \int_0^{a} \cos kn\beta \int_0^{\infty} \frac{\lambda \sin(\lambda \sqrt{a^2 - \beta^2})}{\lambda^4 + c^2 (kn)^8}\, d\lambda,$$

$$G_1^b = \nu G_2^b = \frac{4\nu P k^7 c^2}{\pi^3 a} \sum_{n=1}^{\infty} n^6 S(k, n, a) .$$

To simplify the written solutions, we then allow for the fact that [6]

$$\int_0^{\infty} \frac{\sin ax}{x(x^4 + b^4)} dx = \frac{\pi}{2b^4} \left(1 - \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}}\right) ;$$

$$\int_0^{\infty} \frac{x \sin ax}{x^4 + b^4} dx = \frac{\pi}{2b^2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \sin \frac{ab}{\sqrt{2}} ;$$

$$\sum_{m=1}^{\infty} \frac{\sin mx}{m^3} = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} ; \quad \sum_{m=1}^{\infty} \frac{\sin mx}{m^5} = \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} .$$

As a result we have the following expressions for radial displacement and the internal force factors for the basic state at the point (0; 0)

$$w^b = \frac{2Pc^{-2}}{\pi^2 E h a^2 k^4} \left(\frac{\pi^4 \gamma}{90} - \frac{\pi^2 \gamma^3}{36} + \frac{\pi \gamma^4}{48} - \frac{1}{240} \gamma^{15} - \right.$$

$$\left. - k \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^a \cos kn\beta \exp(-\beta_n) \cos \beta_n \, d\beta \right) ;$$

$$T_1^b = - \frac{2P}{\pi^2 Rcka^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^a \cos kn\beta \exp(-\beta_n) \sin \beta_n \, d\beta ;$$

$$G_1^b = \nu G_2^b = \frac{2\nu P}{\pi^2 k^2 a^2} \left(\frac{\pi^2}{6} \gamma - \frac{\pi}{4} \gamma^2 + \frac{1}{12} \gamma^3 - \right.$$

$$\left. - k \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^a \cos kn\beta \exp(-\beta_n) \cos \beta_n \, d\beta \right) ; \quad T_2^b = 0 ;$$

$$\gamma = ka ; \quad \beta_n = (kn)^2 \sqrt{a^2 - \beta^2} \sqrt{\left(\frac{c}{2}\right)} .$$

According to the synthesis method we supplement these results by the solutions of the equation of the local end effect. Written with respect to the resolution function $F(\alpha, \beta)$ it has the form

$$\frac{\partial^4 F}{\partial \alpha^4} + c^{-2} F = \frac{R^4}{D} p(\alpha; \beta) . \tag{5}$$

As above, we construct the solution of Eq. (5) decaying at $|\alpha| \to \infty$ using the Fourier cosine-transform. Allowing for expansion (3), we find

$$F(\alpha, \beta) = \frac{kR^2}{\pi^2 D} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos kn\beta\right) \int_0^{\infty} \frac{\cos \lambda\alpha \, d\lambda}{\lambda^4 + c^{-2}} .$$

To obtain the components of the internal force factors $T_1^{\mathrm{end}}(\alpha, \beta)$, $T_2^{\mathrm{end}}((\alpha, \beta)$, $G_1^{\mathrm{end}}(\alpha, \beta)$, $G_2^{\mathrm{end}}(\alpha, \beta)$, we calculate the derivatives

$$G_2^{\text{end}} = \nu G_1^{\text{end}} = -\nu \frac{D}{R^2}\frac{\partial^2 F}{\partial \alpha^2};\ T_2^{\text{end}} = -\frac{Eh}{R}F;\ T_1^{\text{end}} = 0.$$

This yields the expressions

$$\begin{aligned} T_2^{\text{end}} &= -\frac{EhkR}{\pi^2 D}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos kn\beta\right) \int_0^{\infty} \frac{\cos \lambda\alpha \, d\lambda}{\lambda^2 + c^{-2}};\ T_1^{\text{end}} = 0; \\ G_2^{\text{end}} &= \nu G_1^{\text{end}} = \frac{\nu k}{\pi^2}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos kn\beta\right) \int_0^{\infty} \frac{\lambda^2 \cos \lambda\alpha}{\lambda^2 + c^{-2}} \, d\lambda, \end{aligned} \tag{6}$$

which correspond to the effect of k radial forces applied in section $\alpha = 0$.

To go over to loading around circular planes, we integrate the right-hand side of expressions (6) with respect to region $\alpha^2 + \beta^2 \le a^2$, and then multiply by $P/(\pi a^2 R^2)$. This transformation with account for the known sum of the series [6]

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin nx = \frac{1}{2}(\pi - x)$$

gives the closed forms of solutions

$$\begin{aligned} T_1^{\text{end}} &= 0;\ T_2^{\text{end}} = -\frac{P}{\pi R a^2}(1 - \exp(-\omega)\cos\omega); \\ G_2^{\text{end}} &= \nu G_1^{\text{end}} = \frac{\nu P c}{\pi a^2} \exp(-\omega) \sin\omega;\ \omega = \frac{a}{\sqrt{2c}}. \end{aligned} \tag{7}$$

Summing up the components of the basic state and local effect, we obtain the formulas

$$T_j = T_j^b + T_j^{\text{end}};\ G_j = G_j^b + G_j^{\text{end}},\ j = \overline{1;2}. \tag{8}$$

for calculating forces and moments at the center of the circular plane of loading.

Then we find the radial displacement of the local effect w^{end}. It is represented by the zeroth harmonics in the expansion of the function $F(\alpha, \beta)$ in terms of the angular coordinate β and under the effect of concentrated forces it has the form

$$w^{\text{end}} = \frac{kR^2}{2\pi^2 D} \int_0^{\infty} \frac{\cos \lambda\alpha}{\lambda^4 + c^{-2}} \, d\lambda.$$

Changing over to circular planes of loading, we find

$$w^{\text{end}} = \frac{2kPR^2}{\pi^3 D a^2} \int_0^a \int_0^{\infty} \frac{\cos \lambda\alpha}{\lambda^4 + c^{-2}} \, d\lambda \sqrt{a^2 - \alpha^2} \, d\alpha.$$

Here the improper integral for λ is tabulated [6]

$$\int_0^{\infty} \frac{\cos \lambda\alpha}{\lambda^4 + b^4} \, d\lambda = \frac{\pi}{2\sqrt{2}\, b^3} \exp\left(-\frac{ab}{\sqrt{2}}\right)\left(\cos\frac{ab}{\sqrt{2}} + \sin\frac{ab}{\sqrt{2}}\right).$$

To integrate with respect to λ we expand the exponential function into the Maclauren series and allow for the fact that the integral

TABLE 1. Dimensionless Values of Tangential Forces and Bending Moments at Different Values of Parameters a and k for Shell with Relative Thickness $h/R = 1/400$

a	k	$\overline{T}_1$	$\overline{T}_2$	$100\overline{G}_1$	$100\overline{G}_2$
	2	55.19	33.88	1.74	5.50
0.1	3	53.15	33.88	1.69	5.34
	4	51.14	33.88	1.64	5.18
	6	47.04	33.88	1.55	4.87
	2	30.14	7.94	0.81	2.71
0.2	3	28.05	7.94	0.76	2.56
	4	26.00	7.94	0.72	2.40
	6	21.91	7.94	0.62	2.09
	2	20.01	3.54	0.52	1.73
0.3	3	17.96	3,54	0.47	1.57
	4	15.92	3,54	0.43	1.42
	6	11.82	3.54	0.33	1.11
	2	14.53	1.99	0.37	1.23
0.4	3	12.48	1.99	0.32	1.07
	4	10.43	1.99	0.28	0.92
	6	6.34	1.99	0.18	0.61

$$\int_0^{\pi/2} \sin^m t \cos^2 t dt = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}+2\right)}$$

is expressed in terms of the gamma-function $\Gamma(z)$ [6]. Then

$$w^{\text{end}} = \frac{kP}{4\pi Eh\sqrt{2\pi c}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(\frac{m+1}{2}\right)}{m!\,\Gamma\left(\frac{m}{2}+2\right)} (ab)^m \times$$

$$\times \left(\cos\frac{m\pi}{4} - \sin\frac{m\pi}{4}\right), \quad b = 1/\sqrt{c}. \tag{9}$$

This expansion converges at any ab, and very quickly when $ab \leq 1$. In this case we can calculate w^{end} with good accuracy using the asymptotic formula

$$w^{\text{end}} = \frac{kPb}{4\sqrt{2}\,\pi Eh}\left(1 - \frac{1}{8}(ab)^2 + \frac{4\sqrt{2}}{45\pi}(ab)^3\right) + O(a^4 b^4).$$

We note that another form of w^{end} presentation is possible. It can be presented in terms of special functions in the form

$$w^{\text{end}} = \frac{kP}{2\pi Eha}(\text{bei}_1(ab) - \text{Re}\, L_1(\sqrt{i}\, ab)).$$

Here $i = \sqrt{-1}$; $\mathrm{bei}_1(z)$, $L_1(z)$ are the Thomson and Struve functions, respectively. The first of them is tabulated in [7]. In the absence of tables for these functions their values can be found by the series

$$\mathrm{bei}_1(ab) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2} ab\right)^{2m+1}}{m!\,(m+1)!} \cos \frac{(2m+1)\pi}{4};$$

$$\mathrm{Re}\, L_1(\sqrt{i}\, ab) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2} ab\right)^{2m+2}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\frac{5}{2}\right)} \cos \frac{\pi(m+1)}{2}.$$

Therefore, expression (9) is in fact reduced to the sum of two known series for special functions.

Using the obtained notions w^{b} and w^{end}, we have the following formula

$$w = w^{\mathrm{b}} + w^{\mathrm{end}}.$$

for the calculation of radial displacements.

We dwell now on the results of calculations. Table 1 gives the dimensionless values of tangential forces $\overline{T}_j = -P^{-1}RT$ and bending moments $\overline{G}_j = P^{-1}G_j$. They are calculated at $R/h = 400$, $\nu = 0.3$, and different values of the parameters a and k. The calculation shows that, in spite of the increase in the total external loading, the internal force factors decrease with an increase in the number of planes. This is especially noticeable at relatively large radii of circular loaded regions. We note that the accuracy of the constructed solutions and, correspondingly, the numerical results given in the table does not differ from the accuracy of the solutions obtained earlier [3, 4] for the cases of loading shells in square planes into which the considered circular loaded regions can be inscribed. This accuracy is characterized in the cited works in detail.

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NOTATION

R, h, radius and thickness of shell; E, ν, elasticity modulus and Poisson coefficient of shell material; p, P, intensity of outer pressure and force applied to one circular region; k, number of radial loads uniformly distributed along contour; x, longitudinal coordinate; $\alpha = x/R$, β, dimensionless longitudinal and circumferential coordinates; r, radius of circular region carrying load P. Subscripts and superscripts: 1, 2, indicate longitudinal and circumferential direction, respectively; b and end indicate basic state and end effect, respectively; m, n, integer indices of summation.

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